Legendrian Contact Homology

Assumptions: working in $\mathbb{R}^3$ (e.g. open unit ball) with contact structure $\alpha = dz + x\,dy$.
(M, ξ) $\alpha = dz - y\,dx$

References:

Introduction:

A knot $K: S^1 \hookrightarrow M$ is a **Legendrian** if $T_pK \subset \xi_p$ $\forall\, p \in K(S^1)$. That is, K is a closed integral curve of the contact structure.

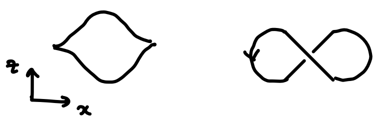
Given any knot K, can ask about how one could classify the knots.

Topologically $\to$ project onto some plane s.t. diagram is nondegenerate

Contactly $\to$ **front projection** $\Pi: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,z)$

Lagrangian projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$

Example: Legendrian unknot:



Care mostly about Lagrangian Projection for today.

Remark: every knot can be represented by a Legendrian

Remark: with our convention, we only allow kinks of the form  or  (anticlockwise goes downstairs).

Motivation:

Remark: any Legendrian has associated to it three classical invariants: its **smooth isotopy class, its rotation number, and** its **Thurston-Bennequin invariant.** Two classification questions are:

1) Which choices of classical invariants can be realized by a Legendrian? (Bennequin Inequality puts restrictions on this)
2) Do there exist L_1, L_2 that share the same classical invariants, but are not Legendrian isotopic?

Eliashberg - Fraser answered 2 as no for the case of smooth unknots.

Chekanov answered this question as yes for other knots:

Chekanov, 2009: for any Legendrian $K \subset \mathbb{R}^3$, **$\exists$ a differential graded algebra**, whose generators are double points of its Lagrangian projection, and the differential is defined combinatorially via the diagram. More generally, one can think of the differential as a count of some J-holomorphic boundary punctured disk.

Idea: build a new invariant, which is now known as the Chekanov-Eliashberg DGA.

Remark: original algebra was for Legendrian knots in $\mathbb{R}^3$, with $\mathbb{Z}_2$ coefficients. Chekanov - Eliashberg (Hofer) then extended the construction to Legendrian submanifolds (with possibly >1 connected components) in an arbitrary contact manifold). (also around the same time). Later, in 2001, Etnyre - Ng - Sabloff lifted the CEDGA to an algebra over $\mathbb{Z}[t, t^{-1}]$, not just $\mathbb{Z}_2$. In the original version, the generators are also relatively $\mathbb{Z}/g$ graded, where g = rotation number of K = rot(K)
In the E-N-S version, the grading lifts to a full $\mathbb{Z}$-grading

Outline of Talk:

1) Define DGA Combinatorially, touching on some of the Floer-Theoretic interpretations.
↳ Algebra, grading, differential, first examples
2) Justify why $\partial^2 = 0$.
3) Describe Chekanov's Example.

Chekanov-Eliashberg DGA:

The Algebra:

Floer Theoretic description:

boundary conditions are Lagrangian.

Let $\mathcal{P} = \{ \gamma : [0,1] \to \mathbb{R}^3 \mid \gamma(0), \gamma(1) \in K \}$. This is an ∞ dimensional

Define an action functional $\mathcal{A}(\gamma) := \int_\gamma \lambda$

For a suitable defn of its derivative, $d\mathcal{A}_\gamma$, we find that γ is a critical point of $d\mathcal{A} \Leftrightarrow \dot{x} = \dot{y} = 0$, i.e. γ is a Reeb chord.

Combinatorial Description:

Definition: Let K be a Legendrian in $\mathbb{R}^3$. Choose a Lagrangian projection of K, and label its double points as $\{a_1, \dots, a_n\}$. closed loop $\Rightarrow$ finitely many double points. These correspond to Reeb chords.

Let $CH = \mathbb{Z}_2 \langle a_1, \dots, a_n \rangle$ be the free algebra generated by $a_1, \dots, a_n$ over $\mathbb{Z}_2$ coefficients.

elements: Words in Reeb chords.

The Grading:

an algebra-generating element inherits a grading that can be described both Floer-Theoretically and combinatorially.

Floer-Theoretic Description:

let $a \in C(K)$, and let x_0, x_1 denote its beginning and end points. Choose a capping path $\gamma : [0,1] \to K$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_0$ (runs from top point to bottom point). Taking the Lagrangian projection, the linearized flow along this closed loop defines a path of Lagrangian subspaces $\Gamma(t)$ in $\mathbb{C}$. Note that this path is not closed, as the start and end arises from a double point. We may close $\Gamma(t)$ as follows. Denote $V_0 = \Gamma(0)$ and $V_1 = \Gamma(1)$, and choose a.c.s J s.t $JV_1 = V_0$. Define a path $\lambda(V_1, V_0)(t) = e^{tJ} V_1$, $t \in [0, \pi/2]$. Concatenating $\Gamma * \lambda$ forms a closed loop in the Lagrangian Grassmannian, and we can therefore associate to it the Maslov Index: $\mu(\Gamma \cdot \lambda(V_1, V_0)) \to$ also called the Conley-Zehnder Index of the path Γ. This is what we define to be $\nu(a)$. This is independent of choice of J, but does depend on homotopy class of path γ: $\nu_{\gamma_1}(a) - \nu_{\gamma_2}(a) = \mu(\gamma_1 * (-\gamma_2))$. So $\nu(a)$ is well-defined modulo Maslov number. We set $|a| = \nu_\gamma(a) - 1$.

Combinatorial Description:

For each double point, the Legendrian K is split into two curves, which we orient as going from the top strand to the bottom strand. Call these two curves C_1 and C_2. Wlog, crossings are orthogonal. Then for each $\varepsilon \in \{1,2\}$ the rotation number is of the form $rot(\pi(C_\varepsilon)) = \frac{1}{2} N_\varepsilon + \frac{1}{4}$ (it is an odd multiple of $\frac{1}{4}$). We set $|a| = N_\varepsilon$ for either ε. Or equivalently, $|a| = 2\,rot(\pi(C_\varepsilon)) - \frac{1}{2}$. Note that $N_1 - N_2$ up to sign is equal to the Maslov number (this is twice the rotation number of K).

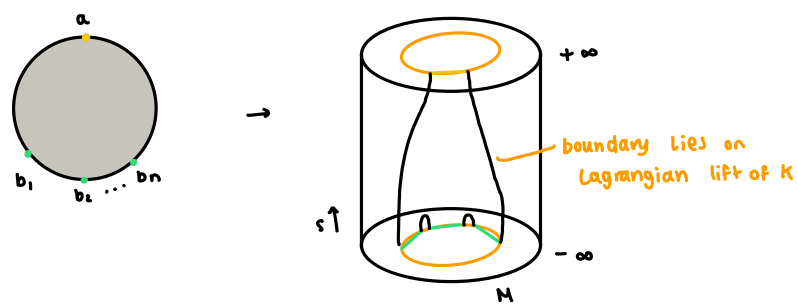
Practically, and for grading lifting reasons, one can choose a basepoint, and define gradings by choosing capping paths that avoid (*).

We extend the grading to words $w \in A$ by letting $|w| = \Sigma$ | letters in w with multiplicity|.

The Differential

Floer Theoretic description:

Roughly speaking, we count punctured pseudoholomorphic disks in the symplectization of M that look like:



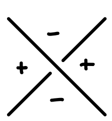
We define the moduli space $\mathcal{M}(a; b_1, \dots, b_n) = \{ J\text{-holo } u : (D_n^2, \partial D_n^2) \to (\mathbb{R}^4, \mathbb{R} \times K) \}$, the parametrized maps that act as above. The dimension of the Moduli space is equal to $\dim(\mathcal{M}) = |a| - \sum_{i=1}^{n} |b_i|$, which in the case that $\dim(\mathcal{M}) = 1$, gives us a well-defined count modulo parametrization.

As usual, we set $\langle \partial a, b \rangle = \# \tilde{\mathcal{M}}(a; b_1, \dots, b_n)$, where $b = b_1 \dots b_n$, and extend linearly.

Morally, we can view the curves under the double projection symplectization $\to$ contact mfld $\to$ Lagrangian projection ($\mathbb{R}^4$, $\mathbb{R}^3$, $\mathbb{R}^2$) and see these disks combinatorially.

Combinatorial Description:

In order to make sense of the orientation of these disks, we decorate each crossing with signs:



(the allocation of signs depend on induced orientation on Lagrangian rel canonical orientation on the complex disk)

Instead of a moduli space of J-curves in the symplectization, we now count:

$$\Delta(a; b_1, \dots, b_n) = \{ u : (D_n^2, \partial D_n^2) \to (\mathbb{R}^2_{xy}, \pi(K)) \text{ satisfying (1) - (4)} \}$$

1) u is an immersion
2) u sends boundary punctures to crossings of $\pi(\Lambda)$
3) u : x to a and a nhood is mapped to a quad of a labelled w/ +ve <u>Reeb</u> sign
   y_i to b_i and a nhood is mapped to a quad of b_i labelled w/ −ve <u>Reeb</u> sign.

Remark: $\Delta(a; b_1, \dots, b_n) \neq \emptyset$ then $|a| - \sum_{i=1}^{n} |b_i| = 1$

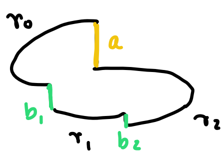
idea: pick a basepoint that does not intersect the boundary of the disk, and compare the (signed) capping paths for the generators. You get the signed difference of paths is exactly the boundary of the disk, which has rotation number 1 (preserved under immersion).

As before, we set $\langle \partial a, b \rangle = \# \tilde{\Delta}(a; b_1, \dots, b_n)$, where $b = b_1 \dots b_n$. and extend linearly.

Remark: ∂ is well-defined, i.e. is a finite sum. To see this, note that if $\exists\, u \in \Delta(a; b_1, \dots, b_n)$, -

$$h(a) - \sum_{i=1}^{n} h(b_i) > 0$$

Consider a closed curve in $\mathbb{R}^3$ comprising Reeb chords and lifts of the boundary components of $u(D^2)$ onto L:



For example, $\Gamma = a \cup \gamma_0 \cup b_1 \cup \gamma_1 \cup b_2 \cup \gamma_2$

Note that the total height gained by walking the full closed loop is 0, i.e. $\int_\Gamma dz = 0$.
We analyse this integral:

$$0 = \int_\Gamma dz = \int_{a, b_1, \dots, b_n} dz + \int_{\gamma_0, \gamma_1, \dots, \gamma_n} dz$$

$$\Rightarrow \int_{a, b_1, \dots, b_n} dz = - \int_{\gamma_0, \gamma_1, \dots, \gamma_n} dz$$

$$= - \int_{\gamma_0, \dots, \gamma_n} z'(t)\, dt$$

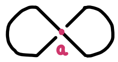
(Since γ lie on K) $= - \int_{\gamma_0, \dots, \gamma_n} x(t) y'(t)\, dt$

(since integral only depends on x, y coordinates) $= - \int_{\pi(\gamma_0, \dots, \gamma_n)} - x(t) y'(t)\, dt$

(By construction) $= \int_{u(\partial D)} x\, dy = \int_{\partial D} u^*(x\, dy) \overset{\text{Stokes'}}{=} \int_D u^*(dx \wedge dy) > 0$. $\square$

Examples

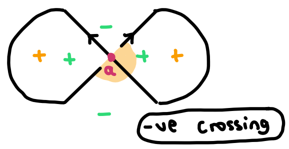
Example 1: the unknot

L = 

algebra: $\mathbb{Z}\langle a \rangle$ → grading: $|a| = 1$

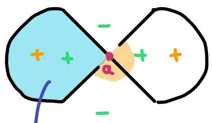
Chains : $\mathbb{Z}\langle 1, a, a^2, \dots, a^n, \dots \rangle$

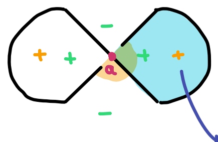
Differential: suffices to compute $\partial(a)$ and extend to rest of generators using (graded) Leibniz :



In the diagram, we've made both Reeb sign and orientation sign assignments.

There are two disks associated to a, both which have −ve assympotic end 1 (so only 1 +ve puncture and no −ve puncture)





D_1 has +ve Reeb sign towards a, and has $\varepsilon(u) = +1$ b.c. not intersecting −ve orientation quad of a, and no other boundary punctures.

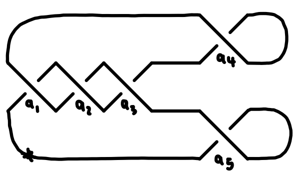
D_2 has +ve Reeb sign towards a, and has $\varepsilon(u) = +1$ b.c. not intersecting −ve orientation quad of a, and no other boundary punctures.

Hence, $\partial(a) = (+1)(1) + (+1)(1) = 2$

Hence, $\partial(a) = 2 \Rightarrow \partial \equiv 2$. ($\equiv 0 \mod 2$)

Legendrian homology : $\mathrm{Ker}(\partial) / \mathrm{Im}(\partial) = \langle 1, a, a^2, \dots, a^n, \dots \rangle / 0 = \langle 1, a, a^2, \dots, a^n, \dots \rangle$.

Example 2: the positive trefoil



$|a_4|, |a_5| = 1$

$|a_1| = |a_2| = |a_3| = 0$.

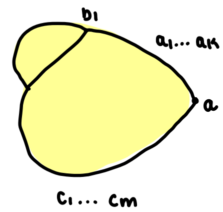
$\partial(a_4)$:

Why $\partial^2 = 0$

When determining ∂, we select

Combinatorially: say

Picture associated:

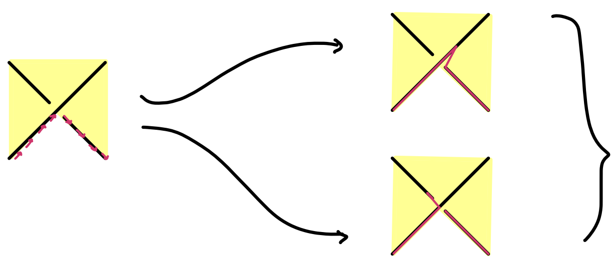


$\partial(a) = a_1 \dots a_k b_1 c_1 \dots c_m$

$\partial^2(a) = \underbrace{a_1 \dots a_k (b_2 \dots b_\ell) c_1 \dots c_m}$

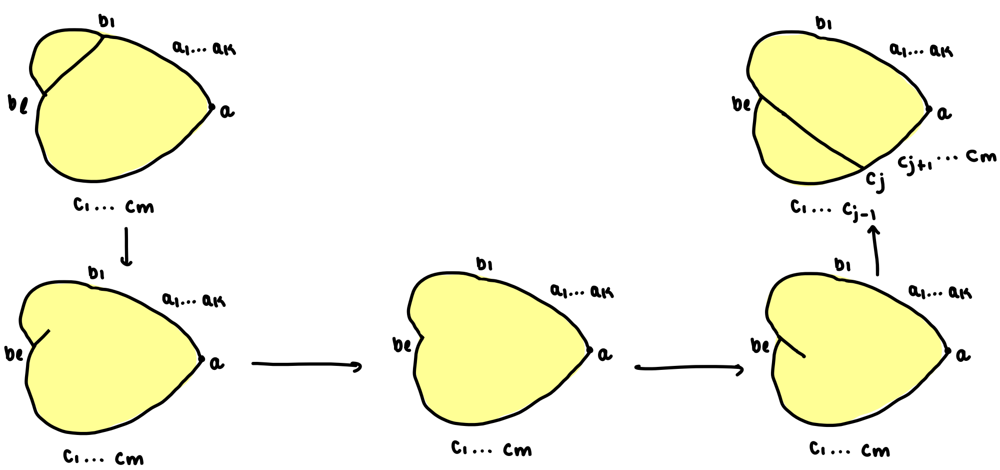
what this term really counts is two disks glued at the crossing b_1

I.e. ∂^2 coefficients count the number of J-holo boundary punctured disks with one obtuse corner. This is a moduli space that is 1 dimensional. Compactifying, we know that boundary points come in cancelling pairs, so that each monomial has a signed count of 0. In the rigid picture, we see the following:



limiting behaviour of these branches show how the disks can degenerate in two different directions.

I.e.



A note on Invariance + General Remarks

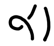
- Two Legendrian isotopic knots have isomorphic dgas in the following sense:

$$\begin{tikzcd} A(K_1) \arrow[d] & A(K_2) \arrow[d] \\ S(A(K_1)) \arrow[r, "(\dagger)"'] & S'(A(K_1)) \end{tikzcd}$$

$(\dagger)$ is a chain isomorphism that is exactly a composition of (finitely many) elementary automorphisms (those that preserve all generators except for one).

- **Cor:** any two Legendrian isotopic knots have the same LCH.

- This is only true in one direction. In particular:

Prop: If K is a stabilized Legendrian knot (add  or ), then the LCH vanishes.

pf: 

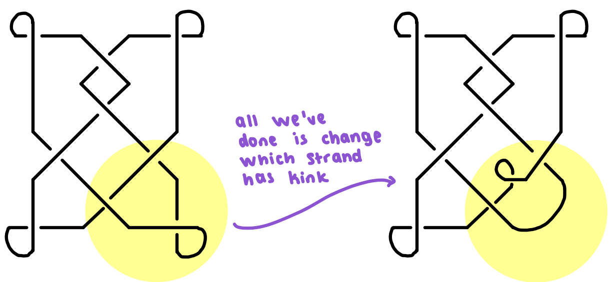
Introduce a new generator a, with area of disk sufficiently small so that $h(a) < h(b)$ $\forall\, b$ double point in original diagram. Then $\partial(a) = 1 \rightarrow$ only count disk in lobe formed. Now for any $w \in \ker(\partial)$, $\partial(aw) = w$ by Leibniz rule $\Rightarrow$ all cycles are boundaries $\Rightarrow$ LCH = 0.

- Vanishing LCH $\not\Rightarrow$ knot is stabilized, example due to Sivek, 2013, knot $m(10_{132})$.

- does not characterize unknot

Proof of Chekanov's Result:

Chekanov's knots:



L_1 L_2

Can read off: $m(L_1) = m(L_2) = 0$, $\beta(L_1) = \beta(L_2) = 1$.

and both isotopic, are 5_2 knots.

One can write down